ON A THEOREM OF HAIMO REGARDING CONCAVE MAPPINGS

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Dedicated to the memory of Jan Krzyż

Abstract. A relatively simple proof is given for Haimo's theorem that a meromorphic function with suitably controlled Schwarzian derivative is a concave mapping. More easily verified conditions are found to imply Haimo's criterion, which is now shown to be sharp. It is proved that Haimo's functions map the unit disk onto the outside of an asymptotically conformal Jordan curve, thus ruling out the presence of corners.

§1. Introduction.

This paper arises from a theorem of Deborah Haimo [10] that gives a sufficient condition for a meromorphic function to map the unit disk $\mathbb{D}$ univalently onto the complement of a bounded convex set. Her theorem involves an estimate of the Schwarzian derivative

$$\mathcal{S}f = (f''/f')' - \tfrac{1}{2}(f''/f')^2$$

and relies upon methods of Nehari [12,13] to prove univalence of analytic functions whose Schwarzian derivative is suitably controlled. Here we give a considerably shorter, more direct proof based essentially on Sturm comparison. In addition, we look more closely at Haimo's criterion, resolve the issue of sharpness, and point out some further consequences.

To set the stage for Haimo's theorem, we need to introduce some terminology. Let $q(x) \geq 0$ be a continuous function defined on $[0,1)$ for which the solution of

$$y''(x) + q(x)y(x) = 0\,, \qquad y(0) = 0\,, \quad y'(0) = 1 \tag{1}$$

is positive on $(0,1)$. The quantity $y'(x)/y(x)$ is thus well defined on the open interval and is easily seen to be decreasing. The function q will be said to belong to the class $P(\alpha)$ if

$$\lim_{x\to 1-} \frac{y'(x)}{y(x)} \geq \alpha \geq 0\,. \tag{2}$$

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Haimo's Theorem. *Let f be a locally univalent meromorphic function in $\mathbb{D}$ with the form*

$$f(z) = \frac{1}{z} + b_0 + b_1 z + b_2 z^2 + \cdots, \qquad z \in \mathbb{D}. \tag{3}$$

If $|\mathcal{S}f(z)| \le 2q(|z|)$ for some $q \in P(\frac{1}{2})$ and all $z \in \mathbb{D}$, then f is univalent in $\mathbb{D}$ and maps it onto the complement of a bounded convex set.

In modern parlance, a meromorphic univalent function is said to be *concave* if its image is the complement of a compact convex set. Prior to Haimo's work, Gabriel [9] had considered the special case where $q(x) \equiv c$ for some constant $c > 0$ and determined the largest value for which the condition $|\mathcal{S}f(z)| \le 2c$ implies that f is a concave mapping. This optimal constant q_0 turned out to be the smallest positive solution of the equation $\tan\sqrt{x} = 2\sqrt{x}$. Simple estimates show that $\pi/3 < \sqrt{q_0} < \pi/3 + \pi/12$, and a numerical calculation gives $q_0 = 1.358\ldots$. For $q(x) \equiv q_0$ the solution to (1) is $y = (1/\sqrt{q_0})\sin\sqrt{q_0}x$, which shows that the constant function $q(x) \equiv q_0$ is of class $P(\frac{1}{2})$.

As in the original paper [10], the strategy of our proof of Haimo's theorem will be to show that f has the property

$$1 + \operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} \le 0, \qquad z \in \mathbb{D}. \tag{4}$$

Because of the simple pole at the origin, this will imply that f maps concentric circles univalently onto convex curves (*cf.* [7], p. 43).

We also give some conditions for a function q to be in the class $P(\alpha)$ for given α. We show that every such function is integrable over $[0,1)$, and that $q \in P(\alpha)$ for any given $\alpha \in [0,1)$ if its integral is less than or equal to $1-\alpha$. In view of Haimo's theorem, it follows that if f has the form (3) and $|\mathcal{S}f(z)| \le 2q(|z|)$ for some continuous function $q(x) \ge 0$ with $\int_0^1 q(x)\,dx \le \frac{1}{2}$, then f is a concave mapping. We will show further that under the same hypotheses f has Schwarzian norm

$$\|\mathcal{S}f\| = \sup_{z\in\mathbb{D}} (1-|z|^2)^2 |\mathcal{S}f(z)| < 2,$$

which gives again a proof of univalence, by a theorem of Nehari [12]. This result may be compared with another theorem of Nehari [14] (also see [3]) that every convex mapping has Schwarzian norm $\|\mathcal{S}f\| \le 2$. For the full class of concave mappings (3), however, nothing better than $\|\mathcal{S}f\| \le 6$ is true, since the mapping onto the complement of a linear segment is concave and has $\|\mathcal{S}f\| = 6$. But the bound $\|\mathcal{S}f\| \le 6$ is known to be valid for all univalent functions, concave or not.

§2. Proof of Haimo's theorem.

We now turn to our simplified proof of Haimo's theorem, appealing to comparison theorems for complex differential equations. By hypothesis, f has the form (3) and $|\mathcal{S}f(z)| \le 2q(|z|)$ for some $q \in P(\frac{1}{2})$. The function $u = (f')^{1/2}$ satisfies

$u'' + \frac{1}{2}(\mathcal{S}f)u = 0$ with $u(0) = 0$ and $|u'(0)| = 1$. For fixed θ, consider the function $v(r) = |u(re^{i\theta})|$. Then v is differentiable away from the zeros of u and we claim that

$$v''(r) + \tfrac{1}{2}|\mathcal{S}f(re^{i\theta})|v(r) \geq 0\,, \qquad 0 < r < 1\,, \tag{5}$$

with initial conditions $v(0) = 0$ and $v'(0) = 1$. A proof of (5) is implicit in Lemma 2 of [6], and also in [4], but details are included here for the sake of completeness. First observe that $v(0) = 0$ and $v(r) > 0$ in some open interval $(0, a)$, where $a \leq 1$. Since $v^2 = u\overline{u}$, we find that

$$v(r)v'(r) = \operatorname{Re}\left\{e^{i\theta}\,\overline{u(re^{i\theta})}\,u'(re^{i\theta})\right\}\,,$$

and so $v'(r) \leq |u'(re^{i\theta})|$ for all $r \in (0, a)$. Another differentiation gives

$$v'(r)^2 + v(r)v''(r) = |u'(re^{i\theta})|^2 + \operatorname{Re}\left\{e^{2i\theta}\,\overline{u(re^{i\theta})}\,u''(re^{i\theta})\right\}\,,$$

which shows that $v'(0) = 1$ and

$$v(r)v''(r) \geq \operatorname{Re}\left\{e^{2i\theta}\,\overline{u(re^{i\theta})}\,u''(re^{i\theta})\right\} = -v(r)^2 \operatorname{Re}\left\{e^{2i\theta}\,\tfrac{1}{2}\mathcal{S}f(re^{i\theta})\right\}\,,$$

in view of the differential equation for u. The inequality (5) now follows for all $r \in (0, a)$, where $v(r) > 0$.

In order to show that $v(r) > 0$ in the whole interval $(0, 1)$, and hence to conclude that (5) holds for $0 < r < 1$, consider the expression $w = v'y - vy'$. By hypothesis,

$$w'(r) = v''(r)y(r) - v(r)y''(r) \geq \left(q(r) - \tfrac{1}{2}|\mathcal{S}f(re^{i\theta})|\right) v(r)y(r) \geq 0$$

wherever $v(r) > 0$. Since $w(0) = 0$, an integration shows that $w(r) > 0$, or $v'(r)/v(r) \geq y'(r)/y(r)$ in $(0, a)$. Choosing $\delta \in (0, a)$ and integrating the last inequality from δ to $r \in (\delta, a)$, we have

$$\frac{v(r)}{y(r)} \geq \frac{v(\delta)}{y(\delta)} \to \frac{v'(0)}{y'(0)} = 1 \qquad \text{as} \quad \delta \to 0\,.$$

Thus $v(r) \geq y(r)$ in $(0, a)$. But $y(r) > 0$ in $(0, 1)$, and so the same is true for v and we see that (5) holds throughout the interval. Consequently, $v'(r)/v(r) \geq y'(r)/y(r)$ for all $r \in (0, 1)$.

Finally, we recall that $v(r) = |u(re^{i\theta})| = |f'(re^{i\theta})|^{-1/2}$ and write

$$\begin{aligned}\frac{v'(r)}{v(r)} &= \frac{\partial}{\partial r}\log|u(re^{i\theta})| = \operatorname{Re}\left\{\frac{\partial}{\partial r}\log u(re^{i\theta})\right\}\\ &= -\frac{1}{2}\operatorname{Re}\left\{\frac{\partial}{\partial r}\log f'(re^{i\theta})\right\} = -\frac{1}{2r}\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\}\,, \qquad z = re^{i\theta}\,.\end{aligned}$$

But $v'(r)/v(r) \geq y'(r)/y(r) \geq \frac{1}{2}$, so it follows that

$$1 + \operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} \leq 1 - |z|\,, \qquad 0 < |z| < 1\,.$$

Since the function on the left-hand side is harmonic, the maximum principle allows the conclusion that f satisfies (4), which completes the proof of Haimo's theorem.

§3. Related criteria for concavity.

In view of Haimo's theorem, it is of interest to study the classes $P(\alpha)$. By definition, $P(\alpha) \subset P(\beta)$ if $\alpha > \beta$, so that $P(\alpha) \subset P(0)$ for all $\alpha > 0$. We begin with the observation that every function of class $P(0)$ is integrable over $(0,1)$. Indeed, for $q \in P(0)$ the solution $y(x)$ of (1) is positive and nondecreasing, and so its integrability on $(0,1)$ depends only its rate of increase as $x \to 1$. Fix any point $a \in (0,1)$ and let $a < x < 1$. Then

$$0 \le y'(x) = y'(a) - \int_a^x q(t)y(t)\,dt \le y'(a) - y(a)\int_a^x q(t)\,dt\,,$$

which shows that $\int_a^x q(t)\,dt \le y'(a)/y(a)$.

The following theorem goes in the converse direction.

Theorem 1. *Let q be a continuous function on $[0,1)$ with $q(x) \ge 0$ and*

$$\int_0^1 q(x)\,dx = c \le 1\,.$$

Then $q \in P(\alpha)$ for $\alpha = 1 - c$, but q need not belong to $P(\alpha)$ for any $\alpha > 1 - c$.

Proof. If y is the solution of (1), then $y(x) > 0$ on some interval $(0,a)$ for $0 < a \le 1$. Since $q(x) \ge 0$, it follows from the differential equation that $y''(x) \le 0$ for $0 < x < a$ and so $y(x) \le x < 1$. Consequently,

$$y'(x) = 1 - \int_0^x q(t)y(t)\,dt \ge 1 - \int_0^1 q(t)\,dt = 1 - c \ge 0\,.$$

This shows that $y(x)$ is nondecreasing as long as it remains positive, which implies that $y(x)$ is positive and nondecreasing throughout the interval $(0,1)$. Therefore,

$$\frac{y'(x)}{y(x)} \ge \frac{1-c}{y(x)} \ge 1 - c\,,$$

so that $q \in P(1-c)$.

To see that the result is best possible, let $0 < a < 1$ and define a continuous function q so that $q(x) = 0$ for $0 \le x \le a$, and q is linear on [a,1] with $q(1) = 2c/(1-a)$, so that $\int_0^1 q(x)\,dx = c$. Because $q(x) \equiv 0$ in $[0,a]$, it follows from (1) that $y(x) = x$ in that interval. Therefore, $a \le y(x) \le 1$ for $x \in (a,1)$, and

$$y'(x) = 1 - \int_0^x q(t)y(t)\,dt = 1 - \int_a^x q(t)y(t)\,dt \le 1 - a\int_a^x q(t)\,dt\,,$$

which implies that

$$\lim_{x\to 1-} \frac{y'(x)}{y(x)} \le \frac{1-ac}{y(1)} \le \frac{1}{a} - c\,.$$

Since the bound can be made arbitrarily close to $1 - c$ by choosing the number a sufficiently near to 1, this proves our assertion that q need not belong to $P(\alpha)$ for any $\alpha > 1 - c$. $\square$

Corollary. *Let q be a continuous function on $[0,1)$ with $q(x) \ge 0$ and*

$$\int_0^1 q(x)\,dx \le \tfrac{1}{2}\,.$$

Let f be a locally univalent meromorphic function of the form (3), *and suppose that $|\mathcal{S}f(z)| \le 2q(|z|)$ for all $z \in \mathbb{D}$. Then f is univalent and maps $\mathbb{D}$ onto the complement of a bounded convex set.*

It turns out that the corollary is best possible in that the conclusion may fail if $\frac{1}{2}$ is replaced by any larger number. As a consequence, we see that Haimo's theorem fails if the function q is required only to be of class $P(\alpha)$ for some $\alpha < \frac{1}{2}$.

Theorem 2. *For each number $c > \frac{1}{2}$ there exist a continuous function q with $q(x) \ge 0$ and*

$$\int_0^1 q(x)\,dx = c\,,$$

and a univalent meromorphic function f of the form (3) *that satisfies $|\mathcal{S}f(z)| \le 2q(|z|)$ for all $z \in \mathbb{D}$ but does not map $\mathbb{D}$ onto the complement of a convex set.*

Corollary. *The constant $\frac{1}{2}$ in Haimo's theorem is best possible. For each $\alpha < \frac{1}{2}$ there exist a function $q \in P(\alpha)$ and a univalent meromorphic function f of form* (3) *that satisfies $|\mathcal{S}f(z)| \le 2q(|z|)$ but does not map $\mathbb{D}$ onto the complement of a convex set.*

Deduction of corollary. According to Theorem 1, a function q as described in Theorem 2, with integral equal to c for $\frac{1}{2} < c \le 1$, must belong to the class $P(\alpha)$ for $\alpha = 1 - c < \frac{1}{2}$.

Proof of theorem. For any constant $c \in [0,1]$ and a positive integer n, let f be a meromorphic function of form (3) with Schwarzian $\mathcal{S}f(z) = 2c(n+1)z^n$, and let $q(x) = c(n+1)x^n$. Then $\int_0^1 q(x)\,dx = c$ and $|\mathcal{S}f(z)| = 2q(|z|)$ for all $z \in \mathbb{D}$. To construct the function f (which is uniquely determined up to an additive constant), let u be the analytic function in $\mathbb{D}$ for which

$$u''(z) + c(n+1)z^n u(z) = 0\,, \qquad u(0) = 0\,, \quad u'(0) = 1\,,$$

and note that $u''(0) = 0$. Then for any point $z_0 \ne 0$ in $\mathbb{D}$, the function

$$f(z) = -\int_{z_0}^{z} u(\zeta)^{-2}\,d\zeta$$

has Schwarzian $\mathcal{S}f(z) = 2c(n+1)z^n$ and is normalized to be of the form (3). We claim that for each $c > \frac{1}{2}$, the function y defined by (1) has the property

$$\lim_{x\to 1-} \frac{y'(x)}{y(x)} < \tfrac{1}{2} \tag{6}$$

when n is sufficiently large. This is intuitively clear by analogy with the construction in the proof of Theorem 1, since the function $q(x) = c(n+1)x^n$ peaks near the point 1 for large n and is qualitatively similar to the function q used in that construction.

Deferring a more detailed verification of (6), we now apply it to show that for n sufficiently large the function f is not a concave mapping. The relation (6) implies that $x_1\, y'(x_1)/y(x_1) < \frac{1}{2}$ for some point $x_1 \in (0,1)$. But $u(x) = y(x)$ for $x \in (0,1)$, so it follows that

$$1 + \frac{x_1 f''(x_1)}{f'(x_1)} = 1 + 2x_1 \frac{y'(x_1)}{y(x_1)} > 0, .$$

Hence for large n the function f fails to satisfy the condition (4) and is not a concave mapping.

On the other hand, the function f is univalent in $\mathbb{D}$ for some $c > \frac{1}{2}$ because it has Schwarzian norm $\|\mathcal{S}f\| < 2$, allowing an application of Nehari's theorem [12]. Indeed, a simple calculation shows that

$$\begin{aligned}\|\mathcal{S}f\| &= 2c(n+1)\sup_{z\in\mathbb{D}}(1-|z|^2)^2|z|^n = 2c(n+1)\max_{0\le t\le 1}(1-t^2)^2 t^n \\ &= 2c(n+1)\left(\frac{4}{n+4}\right)^2 \frac{1}{(1+4/n)^{n/2}} \le \frac{64c}{5^{5/2}} < (1.145)c < 2\end{aligned}$$

for all $c < 0.873$. In particular, $\|\mathcal{S}f\| < 2$ for some $c > \frac{1}{2}$ and all $n \ge 1$. (Curiously, the calculation also shows that $\|\mathcal{S}f\| \to 0$ as $n \to \infty$.)

It remains to verify that (6) holds for n sufficiently large. Let $0 < a < x < 1$. Then for each $\varepsilon > 0$ we have $\int_a^1 q(t)\,dt > (1-\varepsilon)c$ when n is sufficiently large. Hence

$$y'(x) = 1 - \int_0^x q(t)y(t)\,dt \le 1 - a\int_a^x q(t)\,dt \le 1 - a(1-\varepsilon)c\,,$$

so that

$$\lim_{x\to 1-} \frac{y'(x)}{y(x)} \le \frac{1}{a} - (1-\varepsilon)c\,.$$

Thus if $c > \frac{1}{2}$ the bound can be made less than $\frac{1}{2}$ with suitable choice of the parameters a and ε. This verifies (6) and completes the proof of Theorem 2. □

Theorem 3. *Let $q(x) \ge 0$ a continuous function on $[0,1)$ with $\int_0^1 q(x)\,dx \le c$ for some $c \in (0,\infty)$, Let f be a locally univalent meromorphic function whose Schwarzian derivative satisfies $|\mathcal{S}f(z)| \le 2q(|z|)$ for all $z \in \mathbb{D}$. Then $\mathcal{S}f$ is area-integrable over the disk, and f has Schwarzian norm $\|\mathcal{S}f\| < 4c$.*

Proof. Observe first that

$$\iint_{\mathbb{D}} |\mathcal{S}f(z)|\,dxdy \le 2\iint_{\mathbb{D}} q(|z|)\,dxdy = 2\int_0^{2\pi}\int_0^1 q(r)\,r\,drd\theta < 4\pi c\,,$$

where $z = x + iy$. Since the mean-value property of analytic functions gives

$$\mathcal{S}f(0) = \frac{1}{\pi} \iint_{\mathbb{D}} \mathcal{S}f(z)\, dxdy\,,$$

it follows that $|\mathcal{S}f(0)| < 4c$. To obtain a bound at an arbitrary point $\zeta \neq 0$ in $\mathbb{D}$, consider the function

$$g(z) = f(\varphi(z))\,, \qquad \text{where} \ \ \varphi(z) = \frac{z+\zeta}{1+\bar{\zeta}z}\,.$$

Then $\mathcal{S}g(z) = \mathcal{S}f(\varphi(z))\varphi'(z)^2$ and

$$(1-|\zeta|^2)^2|\mathcal{S}f(\zeta)| = |\mathcal{S}g(0)| \leq \frac{1}{\pi}\iint_{\mathbb{D}} |\mathcal{S}g(z)|\, dxdy$$

$$= \frac{1}{\pi}\iint_{\mathbb{D}} |\mathcal{S}f(\varphi(z))||\varphi'(z)|^2\, dxdy = \frac{1}{\pi}\iint_{\mathbb{D}} |\mathcal{S}f(w)|\, dudv < 4c\,,$$

where $w = u + iv$. Hence $\|\mathcal{S}f\| < 4c$. $\square$

Observe that Theorem 3 makes a connection between Haimo's theorem and the Nehari class of functions with $\|\mathcal{S}f\| \leq 2$, guaranteed to be univalent by Nehari's theorem [12]. In fact, for $c \leq \frac{1}{2}$ Theorem 3 gives the stronger conclusion that $\|\mathcal{S}f\| < 2$, and a theorem of Ahlfors and Weill then shows that f maps the disk onto a region outside a bounded Jordan curve which is a quasicircle. By comparison, for $c \leq \frac{1}{2}$ the corollary to Theorem 3 gives (via Haimo's theorem) the further information that the boundary is a convex curve, but this does not imply that $\|\mathcal{S}f\| \leq 2$.

There are many concave mappings that fall outside the jurisdiction of Haimo's theorem. For instance, for $0 < t < 1$ and $s = \sqrt{1+t}$ the function

$$h(z) = \frac{1}{s}\frac{1-w^s}{1+w^s}\,, \qquad \text{where} \ \ w = \frac{1-z}{1+z}\,,$$

which is normalized by $h(0) = 0$ and $h'(0) = 1$, maps the unit disk univalently onto a nonconvex region bounded by two circular arcs meeting with interior angle $s\pi$ (*cf.* [6]). Its inversion $f = 1/h$ has the form (3) and maps the disk onto the complement of a convex set bounded by two circular arcs meeting at interior angle $(2-s)\pi$. Thus f is a concave mapping, but its Schwarzian is $\mathcal{S}f(z) = \mathcal{S}h(z) = -2t(1-z^2)^{-2}$, which has a double pole at the point $z = 1$ and cannot be bounded by an integrable function q as in Haimo's theorem. For the same reason, no mapping onto the exterior of a convex polygon (see [5]) can satisfy the hypotheses of Haimo's theorem.

It would be interesting to find a geometric description of the class of concave mappings described by Haimo's theorem. In the next section we show that the boundary of the omitted set is asymptotically conformal.

§4. Asymptotic Conformality.

Let C be a Jordan curve and let $\gamma(w_1, w_2)$ denote the arc of smaller diameter between the points w_1 and w_2 on C. The curve C is said to be *asymptotically conformal* if

$$\max_{w\in\gamma(w_1,w_2)} \frac{|w_1 - w| + |w - w_2|}{|w_1 - w_2|} \to 1 \qquad \text{as} \quad |w_1 - w_2| \to 0\,.$$

It is clear from this definition that the curve C can not have corners.

We now strengthen Haimo's theorem to show that the function f maps the disk onto the region outside an asymptotically conformal Jordan curve. In particular, the boundary contains no corners, ruling out a convex polygon and the union of two circular arcs, as described at the end of the previous section.

Theorem 4. *Let f be a locally univalent meromorphic function of the form* (3) *with the property that $|\mathcal{S}f(z)| \le 2q(|z|)$ for some $q \in P(\frac{1}{2})$ and all $z \in \mathbb{D}$. Then f is univalent and maps $\mathbb{D}$ onto the complement of a convex set that is either a linear segment or is bounded by an asymptotically conformal Jordan curve.*

In preparation for the proof, it is helpful record a simple lemma. Recall that the Bergman space A^1 consists of all analytic functions g in the disk $\mathbb{D}$ with finite area integral:

$$\iint_{\mathbb{D}} |g(z)|\, dxdy < \infty\,.$$

Let $M_\infty(r, g)$ denote the maximum modulus of $g(z)$ on the circle $|z| = r < 1$.

Lemma. *If $g \in A^1$, then $(1 - r)^2 M_\infty(r, g) \to 0$ as $r \to 1$.*

A proof may be found in [8], p. 79.

Proof of theorem. According to Haimo's theorem, f maps $\mathbb{D}$ univalently onto the complement of a closed bounded convex set. If such a set is not a linear segment, it must be bounded by a Jordan curve C. We are to show that C is asymptotically conformal. As noted at the beginning of Section 3, every function $q \in P(\frac{1}{2})$ is integrable over $(0, 1)$. Hence by Theorem 3, the Schwarzian derivative $\mathcal{S}f$ belongs to the Bergman space A^1, and the lemma then says that

$$(1 - |z|)^2 \mathcal{S}f(z) \to 0 \qquad \text{as} \quad |z| \to 1\,. \tag{7}$$

By a theorem of Pommerenke [11], the boundary curve C is asymptotically conformal if and only if

$$(1 - |z|)\frac{f''(z)}{f'(z)} \to 0 \qquad \text{as} \quad |z| \to 1\,. \tag{8}$$

But Becker and Pommerenke [2] showed that (7) implies (8), so the proof of Theorem 4 is complete. □

It may be remarked that the implication "(8) $\Longrightarrow$ (7)" is a simple consequence of the fact that for any function g analytic in the disk, $(1-|z|)g(z) \to 0$ if and only if $(1-|z|)^2 g'(z) \to 0$. However, the implication "(7) $\Longrightarrow$ (8)" lies deeper because it is not immediately evident that each of the terms in $\mathcal{S}f = (f''/f')' - \frac{1}{2}(f''/f')^2$ has the same order of growth as $\mathcal{S}f$.

§5. Examples.

Theorem 1 allows us to give explicit examples of functions $q \in P(\frac{1}{2})$ for use in Haimo's theorem. The functions

$$q(x) = \frac{1}{\pi\sqrt{1-x^2}}\,, \qquad q(x) = \frac{2}{\pi(1+x^2)}\,,$$
$$\text{and} \quad q(x) = \frac{1-\mu}{2(1-x)^\mu}\,, \qquad \text{where } \ 0 < \mu < 1\,,$$

are all positive and continuous on $[0,1)$, with $\int_0^1 q(x)\,dx = \frac{1}{2}$. Hence $q \in P(\frac{1}{2})$ in each case, by Theorem 1. However, the constants $1/\pi$, $2/\pi$, and $(1-\mu)/2$ cannot be optimal because the distributions of mass in the functions q are not concentrated near the point $x=1$. In order to determine the sharp value in the first case, for instance, one would have to find the solution y of

$$y''(x) + \frac{2c}{\pi\sqrt{1-x^2}}\,y(x) = 0\,, \qquad y(0) = 0\,, \ \ y'(0) = 1\,,$$

then find c by imposing the condition that $y'(x)/y(x) \to 1/2$ as $x \to 1$.

Another example of interest is a mapping onto the outside of an ellipse. For any fixed $r \in (0,1)$, the function $f(z) = 1/z + r^2 z$ has the form (3) and maps the unit disk conformally onto the outside of an ellipse that becomes increasingly eccentric as $r \to 1$. Its Schwarzian derivative is

$$\mathcal{S}f(z) = -\frac{6r^2}{(1-r^2z^2)^2} \qquad \text{and} \qquad \|\mathcal{S}f\| = 6r^2\,, \qquad 0 < r < 1\,.$$

The function $q(x) = 3r^2(1-r^2x^2)^{-2}$ provides the bound $|\mathcal{S}f(z)| \le 2q(|z|)$ and its integral is found to be

$$\int_0^1 q(x)\,dx = \frac{3r}{4}\left\{\frac{2r}{1-r^2} + \log\frac{1+r}{1-r}\right\}\,,$$

a quantity that increases from 0 to ∞ as r goes from 0 to 1. A numerical calculation shows that $\int_0^1 q(x)\,dx = \frac{1}{2}$ for $r = 0.386\ldots$. Thus for $0 < r < 0.386$ the function f satisfies the conditions of Haimo's theorem, as shown by the corollary to Theorem 1. It seems possible that a study of the differential equation $y'' + qy = 0$ would reveal that $q \in P(\frac{1}{2})$ for all $r \in (0,1)$, so that Haimo's theorem always applies, but we have not pursued this question.

§6. A class of Haimo mappings.

In this final section we identify a class of concave mappings, simply described by their Schwarzian derivatives, that are governed by Haimo's theorem.

Theorem 5. *Let f be a concave mapping of $\mathbb{D}$ of the form (3) whose Schwarzian derivative is real and nonnegative on the interval $[0,1)$ and has the property $|\mathcal{S}f(z)| \le \mathcal{S}f(|z|)$ for all $z \in \mathbb{D}$. Then f satisfies the hypotheses of Haimo's theorem.*

Proof. The function $u(z) = -i(f'(z))^{-1/2}$ satisfies $u'' + \frac{1}{2}\mathcal{S}f\, u = 0$, with $u(0) = 0$ and $u'(0) = 1$. By hypothesis, the function $q(x) = \frac{1}{2}\mathcal{S}f(x)$ provides the bound $|\mathcal{S}f(z)| \le 2q(|z|)$ for all $z \in \mathbb{D}$. Thus we need only show that $q \in P(\frac{1}{2})$. Because $\mathcal{S}f(x)$ is real on $[0,1)$, the function u is also real-valued there. Hence u coincides on $[0,1)$ with the solution y of equation (1). But u cannot vanish in $(0,1)$ since the univalent function f is analytic there, and so u must remain positive on $(0,1)$. By hypothesis f is a concave mapping, so the inequality (4) holds for all $z \in \mathbb{D}$. Since $f''/f' = -2u'/u$, it follows from (4) that

$$1 - 2x\frac{y'(x)}{y(x)} = 1 - 2x\frac{u'(x)}{u(x)} \le 0 \qquad \text{for} \quad 0 < x < 1\,,$$

which implies that

$$\lim_{x\to 1-} \frac{y'(x)}{y(x)} \ge \tfrac{1}{2}\,.$$

Thus $q \in P(\frac{1}{2})$ and the proof is complete. □

To give an example of a concave mapping with the properties of Theorem 5, we mention the function $f(z) = \sqrt{c}\cot(\sqrt{c}z)$, studied by Gabriel [9]. Here c is a small positive constant, and $\mathcal{S}f(z) \equiv 2c$. Gabriel showed, and Haimo's theorem confirms, that for sufficiently small c the function f is univalent in $\mathbb{D}$ and maps it onto the complement of a convex set.

Another example appears in the proof of Theorem 2. For any constant $c \in (0, \frac{1}{2})$, a meromorphic function f of form (3) with Schwarzian $\mathcal{S}f(z) = 2c(n+1)z^n$ is a concave mapping of $\mathbb{D}$ that satisfies the requirements of Theorem 5.

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